

Original Article

No-arbitrage conditions and expected returns when assets have different β 's in up and down markets

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ABSTRACT We present a model of expected returns when assets have different β 's in up and down markets. Using no-arbitrage argument, we show that upside and downside β 's are priced separately, and their risk premiums can be expressed in terms of the price and expected payoff of a call and a put option, respectively, on the market index. For the upside β , the higher the price of the call option relative to its expected payoff, the smaller the risk premium; but for the downside β , the higher the price of the put option relative to its expected payoff, the larger the risk premium. Our model provides a useful perspective on what systematic risks are and how they are priced. Empirical evidence shows that contemporaneous stock returns are strongly correlated with downside β 's, but weakly correlated with upside β 's.

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INTRODUCTION

The traditional capital asset pricing models of Sharpe (1964), Lintner (1965) and Black (1972) have ignored asymmetry in systematic risk. In these models, investors are assumed to maximize a utility function that depends on the mean and variance of portfolio returns. In the real world, however, investors are likely to be only concerned about downside variations of their portfolio returns.

Recognizing this, many studies have used some measures of downside risk to explain expected returns. Harlow and Rao (1989), for

example, use asymmetric β 's in empirical tests of mean-semivariance models, a special case of the mean-lower partial moment equilibrium framework developed by Bawa and Lindenberg (1977).

Empirical evidence on whether stocks have different β 's in up and down markets is mixed. Earlier tests including Fabozzi and Francis (1977), Kim and Zumwalt (1979) and Chen (1982) find that most stocks exhibit β 's that are not statistically different in up and down markets. However, Clinebell *et al* (1993) replicate the Fabozzi and Francis

procedures over a different time period and find that the probability of stocks exhibiting different β 's in up and down markets is significantly greater than that caused by random chance.

The length of intervals used in these studies to estimate upside and downside β 's ranges from 6 to nearly 15 years. This is perhaps based on the implicit assumption that β of a stock should only be dependent on the company's fundamentals such as type of business activities and financial leverage, which should be fairly stable. If, however, a stock's upside and downside β 's vary significantly over time, then use of very long estimation intervals may fail to capture any conditional asymmetry in β . The best example of time-varying β asymmetry is perhaps how β seems to depend on past stock performance. DeBondt and Thaler (1985) and Wiggins (1992) show that stocks with poor past performance have larger upside than downside β 's. β asymmetry is also implied in technical analysis when investors use support and resistance levels to assess upside potential and downside risks, typically over short intervals.

A growing body of research has documented the importance of downside β 's in explaining cross-sectional stock returns. Post *et al* (2012) find that the downside β premium is roughly 4–7 per cent per year compared with a premium of 0–3 per cent for regular market β . Similarly, Ang *et al* (2006) report a downside β premium that is approximately 6 per cent per year and is distinct from compensation for regular market β . Other empirical evidence on downside β premium is also reported in Estrada (2007), Pedersen and Hwang (2007) and Kaplanski (2004).

In this article, we present a model of expected returns when asset returns are driven by a market factor but the β coefficient is different depending on whether return on the market factor is positive or negative. On the basis of no-arbitrage argument, we show that upside and downside β 's are priced separately, and their risk premiums are related to the prices and expected payoffs of call and put

options on the market index. Using a methodology similar to the one used by Ang *et al* (2006), we show that stock returns are strongly correlated with downside β 's but weakly correlated with upside β 's.

THE MODEL

Our model assumes that asset returns respond to innovations in a market factor. The response coefficient, or β , can take on one of the two values, depending on whether the innovation in the market factor results in an up market or a down market. An up market is a time period during which the market return is non-negative, and a down market is a period during which the market return is negative. Mathematically, the assumed return-generating process is as follows:

$$R_i = \alpha_i + \beta_{i+} \cdot R_{m+} + \beta_{i-} \cdot R_{m-} + e_i \quad (1)$$

where R_i is the rate of return on the i th asset, α_i is the intercept term and β_{i+} and β_{i-} are the upside and downside β 's, respectively. R_{m+} is the market return when it is non-negative, and 0 otherwise; similarly, R_{m-} is the market return when it is negative, and 0 otherwise. e_i is an idiosyncratic term. In addition to the above return-generating process, we also make the standard assumptions of perfect capital markets, which include unlimited borrowing, zero trading cost and full information to all market participants. Finally, we assume that there are a sufficiently large number of assets with different β 's in up and down markets.

NO-ARBITRAGE CONDITIONS AND EQUILIBRIUM EXPECTED RETURNS

R_i can be rewritten as follows:

$$R_i = (E(R_i) - \beta_{i+} \cdot E(R_{m+}) - \beta_{i-} \cdot E(R_{m-})) + \beta_{i+} \cdot R_{m+} + \beta_{i-} \cdot R_{m-} + e_i \quad (2)$$

where $E(R_i)$ is the expected return on the i th asset, and $E(R_{m+})$ and $E(R_{m-})$ are expected values of R_{m+} and R_{m-} , respectively.

Consider an investment portfolio for which w_i is the dollar amount invested in the i th asset as a percentage of the total equity of the portfolio. The rate of return on this portfolio is:

$$R_p = \sum w_i \cdot (E(R_i) - \beta_{i+} \cdot E(R_{m+}) - \beta_{i-} \cdot E(R_{m-})) + \sum w_i \cdot \beta_{i+} \cdot R_{m+} + \sum w_i \cdot \beta_{i-} \cdot R_{m-} + \sum w_i \cdot e_i \quad (3)$$

If the weights are selected to satisfy the following conditions:

$$\begin{aligned} \sum w_i &\leq 0 \\ \sum w_i \cdot \beta_{i+} &\geq 0 \\ \sum w_i \cdot \beta_{i-} &\leq 0 \\ \sum w_i \cdot e_i &\approx 0 \end{aligned} \quad (4)$$

then the portfolio has no net capital outlay, has non-negative upside β and non-positive downside β . In addition, the portfolio has negligible idiosyncratic risk. For equilibrium and absence of arbitrage opportunities, the following inequality must hold:

$$\sum w_i \cdot (E(R_i) - \beta_{i+} \cdot E(R_{m+}) - \beta_{i-} \cdot E(R_{m-})) \leq 0 \quad (5)$$

It follows that there must exist $\Phi_0 \geq 0$, $\Phi_+ \geq 0$ and $\Phi_- \geq 0$ such that (see Appendix for proof):

$$E(R_i) - \beta_{i+} \cdot E(R_{m+}) - \beta_{i-} \cdot E(R_{m-}) = \Phi_0 - \beta_{i+} \cdot \Phi_+ + \beta_{i-} \cdot \Phi_- \quad (6)$$

or

$$E(R_i) = \Phi_0 - \beta_{i+} \cdot (\Phi_+ - E(R_{m+})) + \beta_{i-} \cdot (\Phi_- + E(R_{m-})) \quad (7)$$

This formulation describes the equilibrium expected return for any asset based on its upside and downside β 's.

In order to determine the values of Φ_0 , Φ_+ and Φ_- , first consider the riskless asset that is not affected by market forces. For such an asset, both upside and downside β 's are 0. It follows immediately that Φ_0 must be equal to the risk-free interest rate, R_f . Next, consider

the market portfolio for which upside and downside β 's are both equal to 1. Let $E(R_m)$ be the expected return on the market portfolio, then the values of Φ_0 , Φ_+ and Φ_- must satisfy:

$$E(R_m) = \Phi_0 - (\Phi_+ - E(R_{m+})) + (\Phi_- + E(R_{m-})) \quad (8)$$

As $E(R_m)$ is equal to the sum of $E(R_{m+})$ and $E(R_{m-})$ by definition, Equation (8) reduces to:

$$\Phi_+ - \Phi_- = R_f \quad (9)$$

Clearly, within this framework, if upside and downside β 's are equal, the equilibrium expected return equation reduces to the traditional Capital Asset Pricing Model (CAPM).

RISK PREMIUMS AND PRICES OF OPTIONS ON THE MARKET INDEX

In order to further interpret the risk premiums associated with upside and downside β 's, consider a one-period, at-the-money call option and a similar put option on the market index. Let C and P be the prices of the call and put options, respectively, and S be the current value of the market index. The return on a long position in the call option is as follows:

$$R_c = \begin{cases} \frac{S}{C} \cdot R_m - 1 & \text{if } R_m \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

The expected return, and upside and downside β 's of this call option are:

$$\begin{aligned} E(R_c) &= \frac{S}{C} \cdot E(R_{m+}) - 1 \\ \beta_+ &= \frac{S}{C} \\ \beta_- &= 0 \end{aligned} \quad (10)$$

Substituting equations in (10) into Equation (7) yields:

$$\Phi_+ = \frac{C}{S} \cdot (1 + R_f) \quad (11)$$

Similarly, the return on the put option is as follows:

$$R_p = -1 \quad \text{if } R_m \geq 0$$

$$= -\frac{S}{P} \cdot R_m - 1 \quad \text{otherwise}$$

The expected return, and upside and downside β 's of this put option are:

$$E(R_p) = -\frac{S}{P} \cdot E(R_{m-}) - 1$$

$$\beta_+ = 0$$

$$\beta_- = -\frac{S}{P} \quad (12)$$

Substituting equations in (12) into Equation (7) yields:

$$\Phi_- = \frac{P}{S} \cdot (1 + R_f) \quad (13)$$

Thus, Equation (7) can be rewritten as:

$$E(R_i) = R_f - \beta_{i+} \cdot \frac{C \cdot (1 + R_f) - S \cdot E(R_{m+})}{S}$$

$$+ \beta_{i-} \cdot \frac{P \cdot (1 + R_f) + S \cdot E(R_{m-})}{S} \quad (14)$$

It can be easily verified that the values of Φ_0 , Φ_+ and Φ_- in Equations (9), (11) and (13) are consistent with put-call parity as they must regardless of the underlying distribution of market returns.

Equation (14) shows that the risk premiums on the upside and downside β 's are related to prices of call and put options on the market index. Notice that $S \cdot E(R_{m+})$ and $-S \cdot E(R_{m-})$ are expected payoffs of the call and put options, respectively. Therefore, the risk premiums associated with the two β 's are proportional to the differences between the prices of the two options compounded at risk-free rate and their expected payoffs at expiration. This is intuitively appealing because the implications are consistent with what we would think. The more expensive the call option is relative to its expected payoff, the more investors value the upside potential in market returns; similarly, the more expensive the put option is relative to its

expected payoff, the more investors are concerned about downside risk.

Since options offer unlimited upside potential and limited downside risk, we expect their prices to exceed expected payoffs. Thus, we expect asset returns to be negatively correlated with upside β 's and positively correlated with downside β 's. However, Cox and Ross (1976) show that the price of an option is equal to the expected payoff on the option under risk-neutral probabilities, discounted at risk-free rate. In other words, Equation (14) can be rewritten as:

$$E(R_i) = R_f - \beta_{i+} \cdot (E'(R_{m+}) - E(R_{m+}))$$

$$+ \beta_{i-} \cdot (-E'(R_{m-}) + E(R_{m-})) \quad (15)$$

where E' denotes expected values under risk-neutral probabilities. If a dollar in an up market is worth less now than a dollar in a down market, then the risk-neutral probabilities for up (down) markets should be less (greater) than real-world probabilities, and asset returns should be positively correlated with both upside and downside β 's.

While there should clearly be a risk premium on the downside β , the relation between expected return and the upside β seems to depend on how investors discount payoffs in positive market states relative to payoffs in negative states, and how strongly they prefer outsized positive outcomes. If investors as a whole act as lottery buyers, then we expect asset returns to be negatively correlated with upside β 's; if investors as a whole are hedgers, then we expect asset returns to be positively correlated with upside β 's. We think it is an empirical question. Our model shows that risk premium on the upside β is related to the difference between the price of a call option on the market index and the option's expected payoff, but it does not predict whether the price of the call option exceeds or falls short of its expected payoff.

How does our model compare with the traditional CAPM? To answer that question, we rewrite Equation (14) using the average of

and the difference between upside and downside β 's. Rearranging the terms and substituting with put-call parity, Equation (14) becomes:

$$E(R_i) = R_f + \frac{\beta_{i+} + \beta_{i-}}{2} \cdot (E(R_m) - R_f) - \frac{\beta_{i+} - \beta_{i-}}{2} \cdot \left(\frac{C \cdot (1 + R_f) - S \cdot E(R_{m+})}{S} + \frac{P \cdot (1 + R_f) + S \cdot E(R_{m-})}{S} \right) \quad (16)$$

When upside and downside β 's are equal, the above equation reduces to CAPM. But, if upside and downside β 's are different, Equation (16) suggests that the traditional CAPM is misspecified. Specifically, the traditional CAPM leaves out a risk premium related to β asymmetry. That risk premium is compensation for assets that bear asymmetrically higher β in down markets than in up markets, and is distinct from compensation for the average β .

TREND VALUE AND VOLATILITY VALUE IN ASSET RETURNS

The return-generating process in Equation (1) indicates that the expected return on an asset can be decomposed into two components. The first component is α , which is the expected return on an asset conditional on market return being zero. α can be viewed as the 'trend value' in the return on an asset that does not depend on market movement. The second component of expected return on an asset comes from the asset's co-movement with the market. Since market return is random, this component can be viewed as the 'volatility value' in the expected return. It follows naturally that in equilibrium an asset with high trend value should have low volatility value. As we shall see below, low volatility value in our model includes having a large downside β relative to upside β .

Notice that Equation (5) can be rewritten as:

$$\sum w_i \cdot \alpha_i \leq 0 \quad (17)$$

Thus, the no-arbitrage condition can be restated as to say that a well-diversified portfolio with non-negative upside β and non-positive downside β must have a non-positive α . The constraint this imposes on the relation between α and the two β 's can be readily seen from Equation (6) which can be rewritten as:

$$\alpha_i = \Phi_0 - \beta_{i+} \cdot \Phi_+ + \beta_{i-} \cdot \Phi_- \quad (18)$$

or,

$$\alpha_i = \Phi_0 - \frac{\beta_{i+} + \beta_{i-}}{2} \cdot (\Phi_+ - \Phi_-) - \frac{\beta_{i+} - \beta_{i-}}{2} \cdot (\Phi_+ + \Phi_-) \quad (19)$$

Substituting Φ_0 , Φ_+ and Φ_- with values from Equations (9), (11) and (13), we get:

$$\alpha_i = R_f - \frac{\beta_{i+} + \beta_{i-}}{2} \cdot R_f - \frac{\beta_{i+} - \beta_{i-}}{2} \cdot \frac{C + P}{S} \cdot (1 + R_f) \quad (20)$$

The above can be interpreted as the equilibrium trade-off between trend value and volatility value in asset returns. There are two sources of volatility value, one from the average of an asset's upside and downside β 's, and the other from the difference between the two β 's. First, as the average β increases, the part of volatility value in the asset's expected return coming from its co-movement with the market increases, so there should be a decrease in the trend value to offset it. The rate of increase in the volatility value as the average β increases is $E(R_m)$, while the rate of corresponding decrease in the trend value is only R_f . Thus, when investors are risk averse, total expected return increases as the average β increases.

The second trade-off is between trend value and the part of volatility value from β asymmetry. Given the average of upside and

downside β 's, as upside β increases and downside β decreases the volatility value in the expected return increases, thus there is an offsetting decrease in the trend value. This trade-off depends on the prices of the two options on the market index. Higher prices of options relative to their expected payoffs imply that investors are more concerned about the asymmetry in their portfolios' returns.

The relationship between α and upside and downside β 's in Equation (20) can explain the result of some previous research on the market-timing and stock selection abilities of mutual fund managers. Henrikson (1984) and Kon (1983) find that mutual funds exhibiting larger upside than downside β 's tend to have smaller α 's. This finding is interpreted by authors as a trade-off between fund managers' timing and selection abilities. In our model, this negative correlation is not coincidental, rather it is a necessary result of market equilibrium, and is fundamentally different from the financial leverage-based explanation offered by Jagannathan and Korajczyk (1986).

DATA AND EMPIRICAL RESULTS

At the beginning of each year from 1985 through 2012, we take all stocks in the Russell 3000 index and estimate upside and downside β 's for each stock using all daily returns in that year. We use Russell 3000 as our universe because it excludes the smallest and most illiquid names that may skew the results. We use a year's worth of daily returns because it gives us enough observations to estimate upside and downside β 's and yet a year is short enough to capture asymmetry in the two β 's. Our methodology is most similar to Ang *et al* (2006) who also use a year's worth of daily returns to estimate upside and downside β 's.

We define an up market as a day when the market return is non-negative, and a down market as a day when the market return is

negative. The market return we use is the equal-weighted average return of all stocks in our sample. We choose an equal-weighted market index both because we think an equal-weighted market index is a more reasonable candidate for the market factor in our model and because it avoids the lead-lag effect of using a value-weighted index to estimate β 's, especially for smaller cap stocks in our sample. However, although the results are not reported here, we did check the robustness of our results using a value-weighted market index when estimating upside and downside β 's. Even though there is a downward bias in β estimates for small cap stocks especially in the earlier part of our sample interval, results are qualitatively similar.

Table 1 reports average annual return, average daily return, along with upside and downside β 's, and number of observations for each of the 16 groups formed by independently ranking stocks into quartiles on upside and downside β 's. We exclude stocks for which we do not have daily returns for the full year. The total number of observations over the 28-year interval is 77 736, or about 2776 stocks in a year.

The average annual return on each of the four upside β groups is 9.18, 13.56, 13.83 and 11.74 per cent, respectively, from the lowest upside β group to the highest; similarly, the average annual return on each of the four downside β groups is 8.27, 11.79, 12.09 and 16.15 per cent, respectively, from the lowest downside β group to the highest. These numbers indicate that downside β 's have stronger correlation with contemporaneous returns than upside β 's. The average returns are monotonically increasing from the lowest downside β group to the highest, and the spread in average return between the highest and the lowest downside β groups is 7.88 per cent per year, with a *t*-statistic of 10.9. The average returns on the four upside β groups are not monotonic and the spread between the highest and the lowest

Table 1: Average annual returns, daily returns, and upside and downside β 's

Rank for upside β	Rank for downside β					
	1 (low)	2	3	4 (high)	All	
1 (low)	$R_{yr}(\%)$	9.0	10.4	9.8	6.6	9.2
	$R_{day}(\%)$	0.039	0.045	0.043	0.043	0.042
	β_+	0.3	0.3	0.2	-0.0	0.2
	β_-	0.4	0.7	1.1	2.0	0.8
	#Obs.	8918	5048	3070	2387	19 423
2		11.5	13.5	14.4	16.7	13.6
		0.048	0.056	0.059	0.065	0.065
		0.7	0.7	0.7	0.7	0.7
		0.4	0.8	1.1	1.8	0.9
		5415	6701	4773	2548	19 437
3		8.7	11.9	14.5	18.5	13.8
		0.042	0.054	0.064	0.077	0.061
		1.1	1.1	1.1	1.2	1.1
		0.3	0.8	1.1	1.8	1.0
		3113	5223	6461	4646	19 443
4 (high)		-4.5	9.9	8.4	17.2	11.7
		0.019	0.045	0.045	0.078	0.059
		2.1	1.8	1.8	2.0	2.0
		-0.0	0.8	1.1	1.9	1.4
		1977	2465	5139	9852	19 433
All		8.3	11.8	12.1	16.2	12.1
		0.040	0.051	0.055	0.072	0.054
		0.7	0.9	1.1	1.4	1.0
		0.3	0.8	1.1	1.9	1.0
		19 423	19 437	19 443	19 433	77 736

Note: For each year from 1985 through 2012, we estimate upside and downside β 's for each stock using daily returns in the year against an equal-weighted index. We then independently rank stocks into four groups on upside and downside β 's. R_{yr} is the total cumulative return, while R_{day} is the arithmetic mean of all daily returns on a stock in the same year as used to estimate β_+ and β_- . The values reported here are averages of all stocks in a group. #Obs. is the number of observations.

upside β groups is 2.56 per cent, with a t -statistic of 3.7.

Average daily return, calculated as the arithmetic mean of each stock's daily returns in a year, exhibits a similar pattern. Stocks in the lowest downside β group have an average daily return of 0.040 per cent, compared with an average daily return of 0.072 per cent for stocks in the highest downside β group. Similarly, stocks in the lowest upside β group have an average daily return of 0.042 per cent, compared with an average daily return of 0.059 per cent for stocks in the highest upside β group.

We report both average annual returns and average daily returns because it is not clear to us which measure of contemporaneous returns is more appropriate for testing our model. This is relevant because different

upside and downside β groups have different levels of volatility, so there could be material differences between average daily returns and average annual returns. While Table 1 shows that the correlation with upside and downside β 's is similar for average annual returns and for average daily returns, there are some differences. In particular, stocks with high upside and downside β 's have considerably higher volatilities, thus their average annual returns are lower than what their average daily returns would suggest. This is simply the effect of compounding.

Not surprisingly, upside and downside β 's are correlated. This can be seen from Table 1 where groups on the diagonal from the upper left to the lower right have more observations than other groups. To find out how strongly upside and downside β 's are correlated with

Table 2: Regressions of contemporaneous stock returns on upside and downside β 's

	Intercept	β_+	β_-	R^2
A Annual returns	6.1 (13.6)	-0.7 (-2.1)	6.6 (18.8)	0.0047 —
B Average daily returns	0.032 (22.1)	0.002 (1.9)	0.020 (17.4)	0.0046 —

Note: For each year from 1985 through 2012, we estimate upside and downside β 's for each stock using daily returns in the year against an equal-weighted index. We then run separate regressions of annual returns and average daily returns on upside and downside β 's. Annual return is the total cumulative return, while average daily return is the arithmetic mean of all daily returns on a stock in a year. The numbers in parenthesis are t -statistics.

stock returns after controlling for each other, we run pooled regressions of returns on the two β 's.

The results are shown in Table 2. When annual returns are used as the dependent variable, the risk premium on the upside β is -0.7 per cent per year, with a t -statistic of -2.1, while the risk premium on the downside β is 6.6 per cent per year, with a t -statistic of 18.8. The positive and very significant risk premium on the downside β indicates that investors demand higher returns for holding stocks with larger downside β 's. This is consistent with the predictions of our model. As we have argued earlier, the risk premium on the upside β depends on the relative influence of speculators and hedgers. The negative risk premium on the upside β indicates that investors are willing to accept lower returns for stocks with higher upside β 's. This suggests that investors as a whole are speculators.

However, when we use average daily returns in the cross-sectional regression, the risk premium on the upside β becomes positive 0.002 per cent, with a t -statistic of 1.9, while the risk premium on the downside β is 0.020 per cent, with a t -statistic of 17.4. These results indicate that while investors clearly demand higher returns for stocks with larger downside β 's, their aversion to upside β is unclear. This is similar to the findings of Post *et al* (2012), Ang *et al* (2006) and others.

Our model predicts that the intercept of the regressions in Table 2 should be equal to the risk-free interest rate. The intercept is 6.1 per cent per year in the regression of annual returns, and 0.032 per cent per day in the regression of daily stock returns. Similar to Black *et al* (1972) who tested CAPM, the intercept from our regressions here is also higher than the average interest rate on Treasury Bills during the sample interval.

CONCLUSIONS

In this article, we present a model of expected returns when assets have different β 's in up and down markets. Using no-arbitrage argument similar to Ross (1976), we show that upside and downside β 's are priced separately, and their risk premiums can be expressed in terms of the price and expected payoff of a call and a put option, respectively, on the market index. For the upside β , the higher the price of the call option relative to its expected payoff, the smaller the risk premium; but for the downside β , the higher the price of the put option relative to its expected payoff, the larger the risk premium.

The empirical test of the model utilizes a two-pass regression technique similar to that used in previous studies of asset pricing models. On the basis of stock return data from 1985 through 2012, we show that stock returns are positively correlated with downside β 's but weakly correlated with upside β 's. Over the 28-year interval, the average risk premium on the downside β is 6.6 per cent per year when we use annual stock returns in the regression, and is 0.020 per cent per day when using daily stock returns. The risk premium on the upside β is much smaller in magnitude, and negative when using annual returns but slightly positive when using daily returns.

We think our model has the potential to explain some market anomalies that regular market β has failed to explain, but its success of doing that will depend critically on getting good estimates of upside and downside β 's.

This will not be an easy task because we believe upside and downside β 's may be unstable. In fact, the relation between past momentum and downside risk found in previous studies is an indication that stocks' asymmetric risks shift significantly over time. This is consistent with Ang *et al* (2006) who find that past downside β is a poor predictor of future downside β , and may also explain the limited ability of asymmetric β 's in explaining the momentum effect in Ang *et al* (2001).

One possible way to obtain good estimates of real-time upside and downside β 's is using high-frequency return data and correlation matrix implied in stock option prices. Despite their popularity with some practitioners, these data sources have not been widely used in tests of asset pricing models, but may lead us to better understand the dynamic market mechanisms that drive risks and returns.

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APPENDIX

Proof of Equation (7)

Let n be the number of assets, and define a $3 \times n$ matrix B , and vectors w and π as follows:

$$B = \begin{bmatrix} -1 & -1 & \dots & -1 \\ \beta_{1+} & \beta_{2+} & \dots & \beta_{n+} \\ -\beta_{1-} & -\beta_{2-} & \dots & -\beta_{n-} \end{bmatrix}$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix}$$

$$\pi = \begin{bmatrix} -E(R_1) + \beta_{1+} \cdot E(R_{m+}) + \beta_{1-} \cdot E(R_{m-}) \\ -E(R_2) + \beta_{2+} \cdot E(R_{m+}) + \beta_{2-} \cdot E(R_{m-}) \\ \vdots \\ -E(R_n) + \beta_{n+} \cdot E(R_{m+}) + \beta_{n-} \cdot E(R_{m-}) \end{bmatrix}$$

Then the no-arbitrage conditions contained in Equations (4) and (5) can be rewritten as:

$$\text{If } B \cdot w \geq 0 \text{ then } \pi \cdot w \geq 0 \quad (\text{A.1})$$

The problem here is to prove that there must exist a non-negative vector Φ such that

$$\pi = \Phi \cdot B \quad (\text{A.2})$$

The proof is similar to that of Varian (1987).

Consider the following linear programming problem (the primal):

$$\begin{aligned} &\text{Min } \pi \cdot w \\ &\text{s.t. } B \cdot w \geq 0 \end{aligned}$$

where the components of w can be either positive or negative and are unbounded. Certainly, $w = 0$ is a finite and feasible solution to the above primal, and the statement in (A.1) implies that $w = 0$ indeed minimizes the objective function.

The dual of the above primal problem is:

$$\begin{aligned} &\text{Max } \Phi \cdot 0 \\ &\text{s.t. } \Phi \cdot B = \pi \\ &\text{and } \Phi \geq 0 \end{aligned}$$

Since the primal has a finite and feasible solution, so must the dual. Substitution with the definitions of B , w and π yields the following:

$$\begin{aligned} &\begin{bmatrix} -E(R_1) + \beta_{1+} \cdot E(R_{m+}) + \beta_{1-} \cdot E(R_{m-}) \\ -E(R_2) + \beta_{2+} \cdot E(R_{m+}) + \beta_{2-} \cdot E(R_{m-}) \\ \vdots \\ -E(R_n) + \beta_{n+} \cdot E(R_{m+}) + \beta_{n-} \cdot E(R_{m-}) \end{bmatrix} \\ &= [\Phi_0 \Phi_+ \Phi_-] \begin{bmatrix} -1 & -1 & \dots & -1 \\ \beta_{1+} & \beta_{2+} & \dots & \beta_{n+} \\ -\beta_{1-} & -\beta_{2-} & \dots & -\beta_{n-} \end{bmatrix} \end{aligned}$$

or

$$E(R_i) = \Phi_0 - \beta_{i+} \cdot (\Phi_+ - E(R_{m+})) + \beta_{i-} \cdot (\Phi_- + E(R_{m-}))$$

for $i = 1, 2, \dots, n$



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